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On the integrability of the generalized Fisher-type nonlinear diffusion equations

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Abstract
In this paper, the geometric integrability and Lax integrability of the generalized Fisher-type nonlinear diffusion equations with modified diffusion in (1+1) and (2+1) dimensions are studied by the pseudo-spherical surface geometry method and prolongation technique. It is shown that the (1+1)-dimensional Fisher-type nonlinear diffusion equation is geometrically integrable in the sense of describing a pseudo-spherical surface of constant curvature -1 only for $m = 2$, and the generalized Fisher-type nonlinear diffusion equations in (1+1) and (2+1) dimensions are Lax integrable only for $m = 2$. This paper extends the results in Bindu *et al* 2001 (*J. Phys. A: Math. Gen.* **34** L689) and further provides the integrability information of (1+1)- and (2+1)-dimensional Fisher-type nonlinear diffusion equations for $m = 2$.

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1. Introduction

The study of integrability and exact solutions for reaction–diffusion systems has been one of the most challenging problems in recent years. Bindu *et al* [1] considered the Fisher-type reaction–diffusion equation with quadratic nonlinearity and modified diffusion,

$$u_t - \triangle u - \frac{m}{1-u}(\nabla u)^2 - u(1-u) = 0, \qquad m \neq 0, \tag{1}$$

where $u = u(t, x)$ or $u(t, x, y)$ is a certain kinetic variable, $\triangle$ and ∇ are Laplacian and gradient operators, respectively. This is an important physical system appearing in many areas of physics and biology [2–4]. Bindu *et al* pointed out that equation (1) with $m = 2$

is Painlevé integrable [5] for both the (1+1) and (2+1) dimensions. More interestingly, they found that the Bäcklund transformation deduced from the Laurent expansion gives rise to the linearizing transformation in a natural way. Similarly, they showed that a Lie symmetry analysis singles out the $m = 2$ case in equation (1) as the only system possessing a nontrivial infinite-dimensional Lie algebra of symmetries and that the similarity variables and similarity reductions gave rise to the linearizing transformation and several physically interesting solutions, including the travelling wave solutions, static structures and so on known in the literature, in an automatic way.

In this short paper, based on the geometric notion of a differential system describing pseudo-spherical surfaces [6] and the prolongation technique [7, 8], we will give the further integrability information for equation (1) with $m = 2$. We point out that the (1+1)-dimensional case of equation (1) for $m = 2$ describes a pseudo-spherical surface of constant curvature -1, and the (1+1)- and (2+1)-dimensional cases of equation (1) for $m = 2$ are Lax integrable.

2. The (1+1)-dimensional case of equation (1)

It is well known that a differential equation for a real-valued function $u = u(x, t)$ is said to describe pseudo-spherical surfaces (PSSs), i.e. it is geometrically integrable [6] if it is the necessary and sufficient condition for the existence of smooth real functions f_{ij}, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 2$, depending only on u and a finite number of derivatives, such that the 1-forms $\omega_i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$, $1 \leqslant i \leqslant 3$, satisfy the structure equations of a surface of constant Gaussian curvature -1, that is,

$$\mathrm{d}\omega_1 = \omega_3 \wedge \omega_2, \qquad \mathrm{d}\omega_2 = \omega_1 \wedge \omega_3, \qquad \mathrm{d}\omega_3 = \omega_1 \wedge \omega_2. \tag{2}$$

One can verify straightforwardly that (2) is equivalent to saying that

$$\mathrm{d}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \Omega\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \qquad \Omega = \frac{1}{2}\begin{pmatrix}\omega_2 & \omega_1 - \omega_3\\ \omega_1 + \omega_3 & -\omega_2\end{pmatrix} \in sl(2, R) \tag{3}$$

is a completely integrable system, i.e. $\mathrm{d}\Omega - \Omega \wedge \Omega = 0$.

In what follows, we prove that the (1+1)-dimensional case of (1)

$$u_t - u_{xx} - \frac{m}{1-u}u_x^2 - u + u^2 = 0 \tag{4}$$

describes pseudo-spherical surfaces and thus it is geometrically integrable only for $m = 2$. To do so, first let $u_x = p$, then (4) becomes

$$u_t - p_x - \frac{m}{1-u}p^2 - u + u^2 = 0,$$

which can be represented by the set of 2-forms as follows:

$$\begin{aligned}
&\alpha_1 = \mathrm{d}u \wedge \mathrm{d}t + p\,\mathrm{d}t \wedge \mathrm{d}x,\\
&\alpha_2 = \mathrm{d}u \wedge \mathrm{d}x + \mathrm{d}p \wedge \mathrm{d}t + \left(u^2 - u - \frac{m}{1-u}p^2 +\right)\mathrm{d}t \wedge \mathrm{d}x,
\end{aligned} \tag{5}$$

which constitutes a closed ideal $I = \{\alpha_1, \alpha_2\}$.

Following the procedure of Chern and Tenenblat [6], we set $f_{21} = \eta$ then (2) becomes

$$\begin{cases}
\mathrm{d}f_{11} \wedge \mathrm{d}x + \mathrm{d}f_{12} \wedge \mathrm{d}t + (\eta f_{32} - f_{22}f_{31})\,\mathrm{d}x \wedge \mathrm{d}t = 0,\\
\mathrm{d}f_{22} \wedge \mathrm{d}t + (f_{12}f_{31} - f_{11}f_{32})\,\mathrm{d}x \wedge \mathrm{d}t = 0,\\
\mathrm{d}f_{31} \wedge \mathrm{d}x + \mathrm{d}f_{32} \wedge \mathrm{d}t + (\eta f_{12} - f_{22}f_{11})\,\mathrm{d}x \wedge \mathrm{d}t = 0,
\end{cases} \tag{6}$$

where $f_{ij} = f_{ij}(u, p)$ and $\mathrm{d}f_{ij} = f_{ij,u}\,\mathrm{d}u + f_{ij,p}\,\mathrm{d}p$, from which we have

$$\begin{aligned}
&f_{11,p} = f_{22,p} = f_{31,p} = 0, \qquad f_{31,u} - f_{32,p} = 0,\\
&f_{11,u} - f_{12,p} = 0, \qquad f_{11,u}\left(u^2 - u - \frac{mp^2}{1-u}\right) + pf_{12,u} + \eta f_{32} - f_{22}f_{31} = 0, \\
&pf_{22,u} + f_{12}f_{31} - f_{11}f_{32} = 0, \qquad f_{31,u}\left(u^2 - u - \frac{mp^2}{1-u}\right) + pf_{32,u} + \eta f_{12} - f_{22}f_{11} = 0.
\end{aligned} \tag{7}$$

After some computation, we find that (7) has a nontrivial solution only for $m = 2$, i.e.

$$\begin{aligned}
&f_{11} = \frac{(C_2 - C_1 + C_1 u)\eta^2 + C_2 u}{(u-1)\eta(\eta^2+1)}, \qquad f_{32} = \frac{C_2 u_x}{(u-1)^2\eta} - C_1 - \frac{C_2}{u-1},\\
&f_{22} = \eta^2 + 1, \qquad f_{31} = -\frac{(C_2 - C_1 + C_1 u)\eta^2 + C_2 u}{\eta(\eta^2+1)(u-1)}, \\
&f_{12} = -\frac{C_2 u_x}{(u-1)^2\eta} + C_1 + \frac{C_2}{u-1},
\end{aligned} \tag{8}$$

where C_1, C_2 are constants. Therefore, we have found the nontrivial 1-forms

$$\begin{aligned}
\omega_1 &= \frac{(C_2 - C_1 + C_1 u)\eta^2 + C_2 u}{(u-1)\eta(\eta^2+1)}\,\mathrm{d}x + \left[C_1 + \frac{C_2}{u-1} - \frac{C_2 u_x}{(u-1)^2\eta}\right]\mathrm{d}t,\\
\omega_2 &= \eta\,\mathrm{d}x + (\eta^2+1)\,\mathrm{d}t, \\
\omega_3 &= -\frac{(C_2 - C_1 + C_1 u)\eta^2 + C_2 u}{\eta(\eta^2+1)(u-1)}\,\mathrm{d}x + \left[\frac{C_2 u_x}{(u-1)^2\eta} - C_1 - \frac{C_2}{u-1}\right]\mathrm{d}t,
\end{aligned} \tag{9}$$

such that (2) is satisfied, which denotes that the (1+1)-dimensional case of (1), i.e. (4) describes a PSS and thus is geometrically integrable only for $m = 2$.

In what follows, we prove that equation (4) is also Lax integrable by the prolongation technique [6]. To do so, we introduce the system of 1-forms

$$\omega^i = \mathrm{d}y^i - F^i(u, p, y^i)\,\mathrm{d}x - G^i(u, p, y^i)\,\mathrm{d}t, \tag{10}$$

where y^i $(i = 1, 2, \ldots, n)$ are called pseudo-potentials and assume F^i and G^i are of the form $F^i = F^i_j y^j$, $G^i = G^i_j y^j$. For simplicity, we write F^i_j to be F and G^i_j to be G.

We need $I \bigcup \{\omega^i\}$ to be a closed ideal, namely

$$\mathrm{d}\omega^i = \sum_{j=1}^{2} f^i_j \alpha_j + \eta^i \wedge \omega^i, \tag{11}$$

where f^i_j are 0-form and η^i are 1-form, which leads to differential equations for F and G

$$F_p = 0, \qquad F_u = G_p, \qquad p^2\frac{m}{1-u}F_u + uF_u - u^2F_u - pG_u + [F, G] = 0, \tag{12}$$

where $[F, G] = FG - GF$.

The general solution for equation (12) is

$$\begin{aligned}
F &= X_1 + X_2(u-1)^{1-m},\\
G &= \frac{(1-m)p(u-1)^{1-m}}{u-1}X_2 + (u-1)^{1-m}X_3 + X_4,
\end{aligned}$$

where

$$\begin{aligned}
&(u-1)^{1-m}([X_2, X_3](u-1)^{1-m} + [X_1, X_3] + [X_2, X_4])\\
&\qquad + (u-1)^{1-m}u(m-1)X_2 + [X_1, X_4] = 0, \qquad X_3 = [X_1, X_2],
\end{aligned} \tag{13}$$

where X_1, X_2, X_3, X_4 are constant matrices.

Only when $1 - m = \pm 1$, i.e. when $m = 0 \text{or} 2$, one can find the nontrivial X_1, X_2, X_3, X_4 from (13). Because $m \neq 0$, the only case is $m = 2$, so we have

$$\begin{aligned} F &= X_1 + X_2 (u-1)^{-1}, \\ G &= -\frac{X_2}{(u-1)^2} p + \frac{X_3}{u-1} + X_4, \end{aligned} \tag{14}$$

with the commutation relations of X_1, X_2, X_3, X_4 (i.e. prolongation algebra)

$$\begin{aligned} &X_2 + [X_2, X_4] + [X_1, X_3] = 0, \qquad [X_2, X_3] = 0, \qquad [X_1, X_4] + X_2 = 0, \\ &[X_1, X_2] = X_3. \end{aligned} \tag{15}$$

This prolongation algebra has a nontrivial 2×2 matrix representation, so we have two pseudo-potentials y^1, y^2, i.e. $n = 2$. From this representation and (13) and (14), F and G are determined directly as follows:

$$F = \begin{pmatrix} 1 + \frac{\lambda}{u-1} & -\frac{\lambda u}{u-1} \\ \frac{\lambda u}{u-1} & 1 + \frac{\lambda - 2\lambda u}{u-1} \end{pmatrix}, \qquad G = \begin{pmatrix} 1 + \lambda - \frac{\lambda u_x}{(u-1)^2} & \frac{\lambda u_x}{(u-1)^2} \\ 1 - \frac{\lambda u_x}{(u-1)^2} & \lambda + \frac{\lambda u_x}{(u-1)^2} \end{pmatrix}. \tag{16}$$

So the (1+1)-dimensional case of (1) is Lax integrable only for $m = 2$ and the Lax pair in a matrix form is

$$y_x = F y, \qquad y_t = G y, \qquad \text{where} \quad y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, \tag{17}$$

where F and G are given by (16).

3. The (2+1)-dimensional case of equation (1)

In this section, we will study the Lax integrablility of equation (1) in (2+1) dimensions,

$$u_t - u_{xx} - u_{yy} - \frac{m}{1-u}\left(u_x^2 + u_y^2\right) - u + u^2 = 0, \tag{18}$$

which is equivalent to the following system by introducing new dependent variables p, q,

$$\begin{cases} p = u_x, \qquad q = u_y, \qquad p_y = q_x, \\ u_t = p_x + q_y + \dfrac{m}{1-u}(p^2 + q^2) + u - u^2. \end{cases} \tag{19}$$

Following the procedure of [8], assume (18) has such a Lax pair

$$\begin{cases} \xi_x = F\xi + A\xi_y, \\ \xi_t = G\xi + B\xi_y, \end{cases} \tag{20}$$

where ξ is a column vector, A, B are constant matrices, and F, G are matrices with respect to (u, p, q). The integrability condition $\xi_{xt} = \xi_{tx}$ denotes

$$\begin{cases} [A, B] = 0, [A, G] - [B, F] = 0, \\ F_t - G_x + AG_y - BF_y + [F, G] = 0. \end{cases} \tag{21}$$

From (21) we have the matrix algebra-differential equations about A, B and F, G,

$$\begin{aligned} &[A, B] = 0, \quad [A, G] - [B, F] = 0, \quad F_p = F_q = 0, \quad F_u = G_p, \quad F_u + AG_q = 0, \\ &AG_p = G_q, \quad \left[\frac{m}{1-u}(p^2 + q^2) + u - u^2\right] F_u - pG_u + qAG_u - qBF_u + [F, G] = 0. \end{aligned} \tag{22}$$

Using the same procedure as that for (12), we find that (22) has nontrivial solutions F, G and A, B only for $m = 2$, they are

$$A = \begin{pmatrix} 1-\mathrm{i} & 1 \\ -1 & -1-\mathrm{i} \end{pmatrix}, \qquad B = -2A, \qquad \mathrm{i} = \sqrt{-1},$$
$$F = \begin{pmatrix} 2 + \dfrac{\lambda}{u-1} & 1 + \dfrac{\lambda}{u-1} \\ -\dfrac{\lambda}{u-1} & 1 - \dfrac{\lambda}{u-1} \end{pmatrix}, \tag{23}$$
$$G = \begin{pmatrix} \dfrac{2u\lambda - 2u - 3\lambda + 2}{u-1} - \dfrac{\lambda\gamma}{(u-1)^2} & \dfrac{u\lambda - 2u - 2\lambda + 2}{u-1} - \dfrac{\lambda\gamma}{(u-1)^2} \\ \dfrac{\lambda(3u - u^2 - 2 + \gamma)}{(u-1)^2} & \dfrac{\lambda(u - 1 + \gamma)}{(u-1)^2} \end{pmatrix}, \qquad \gamma = u_x - \mathrm{i}u_y.$$

Therefore, the (2+1)-dimensional case of (1) is Lax integrable only for $m = 2$ and the Lax pair is given by (20) with (23) and $\xi = (\xi^1, \xi^2)^T$, where T denotes the transpose of a vector.

4. Conclusions

In summary, by using the PSS geometry method and prolongation technique we have investigated the generalized Fisher-type nonlinear diffusion equations. As a result, we point that the (1+1)-dimensional generalized Fisher-type nonlinear diffusion equation is geometrically integrable in the sense of describing a PSS of constant curvature -1 and is also Lax integrable in the sense of having Lax pair only for $m = 2$. The (2+1)-dimensional generalized Fisher-type nonlinear diffusion equation is Lax integrable only for $m = 2$. These results provide strong information on the integrability of (1+1)- and (2+1)-dimensional Fisher-type nonlinear diffusion equations (1) for $m = 2$. The further work is regarding how to derive interesting solutions of these integrable equations from their given Lax pairs.

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